2D Backward Stochastic Navier-Stokes Equations with Nonlinear Forcing *

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January 20, 2013

Abstract

The paper is concerned with the existence and uniqueness of a strong solution to a two-dimensional backward stochastic Navier-Stokes equation with nonlinear forcing, driven by a Brownian motion. We use the spectral approximation and the truncation and variational techniques. The methodology features an interactive analysis on basis of the regularity of the deterministic Navier-Stokes dynamics and the stochastic properties of the Itô-type diffusion processes.

Keywords: Navier-Stokes equation, backward stochastic equation, adapted solution, existence, uniqueness

AMS Subject Classification: 60H15, 35R60, 35R15, 76D05, 76M35

*Supported by NSFC Grant #10325101, by Basic Research Program of China (973 Program) Grant # 2007CB814904, by the Science Foundation of the Ministry of Education of China Grant #200900071110001, and by WCU (World Class University) Program through the Korea Science and Engineering Foundation funded by the Ministry of Education, Science and Technology (R31-2009-000-20007). Part of this study was reported by the second author at the third International Symposium of Backward Stochastic Differential Equations and their Applications, held in 2002 in Weihai, Shandong Province, China. The third author thanks the School of Mathematical Sciences, Fudan University for the hospitality during his visit in the winter of 2009.

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1 Introduction

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ be a complete, filtrated probability space, on which defined is a standard 1-dimensional Brownian motion $\{W_t\}_{t\geq 0}$, whose natural augmented filtration is denoted by $\{\mathscr{F}_t, t \in [0,T]\}$. We denote by $\mathcal{P}$ the σ-Algebra of the predictable sets on $\Omega \times [0,T]$ associated with $\{\mathscr{F}_t\}_{t\geq 0}$. The expectation will be exclusively denoted by E and the conditional expectation on $\mathscr{F}_s$ will be denoted by $E_{\mathscr{F}_s}$. We use *a.s.* to denote that an equality or inequality holds almost surely with respect to the probability measure P.

The theory of backward stochastic differential equations (BSDEs) has received an extensive studies in the last two decades in connection with a wide range of applications as in stochastic control theory, econometrics, mathematical finance, and nonlinear partial differential equations. See [2–4, 7, 8, 11, 22] for details. In Tang [18], a very general system of backward stochastic partial differential equations (BSPDEs) are studied. However, they are semi-linear, and thus exclude the nonlinearity of the well-known Navier-Stokes operator. In this paper, we concentrate our attentions to study the backward stochastic Navier-Stokes equation (BSNSE).

The standard deterministic Navier-Stokes equation describing the velocity field of an incompressible, viscous fluid motion in a domain of $\mathbb{R}^d$ ($d = 2$ or 3) takes the following form:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p + f = 0, \quad t \geq 0; \\ \nabla \cdot u = 0, \quad u(0) = u_0, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ represents the d-dimensional velocity field of a fluid, $p = p(t,x)$ is the pressure, $\nu \in (0,\infty)$ is the viscosity coefficient, and $f = f(t,x)$ stands for the external force. Let (u,p) solve the equation (1.1). By reversing the time and defining

$$\tilde{u}(t,x) = -u(T-t,x), \quad \tilde{p}(t,x) = p(T-t,x), \quad \text{for } t \leq T,$$

then $(\tilde{u},\tilde{p})$ satisfies the following *backward* Navier-Stokes equation:

$$\begin{cases} \partial_t \tilde{u} + \nu\Delta \tilde{u} + (\tilde{u}\cdot\nabla)\tilde{u} + \nabla \tilde{p} + f = 0, \quad t \leq T; \\ \nabla \cdot \tilde{u} = 0, \quad \tilde{u}(T) = \tilde{u}_0. \end{cases} \tag{1.2}$$

Note that the time-reversing makes the original initial value problem of (1.1) become a terminal value problem of (1.2).

We shall study the following two-dimensional backward stochastic Navier-Stokes equations (briefly 2D BSNSE) in $\mathbb{R}^2$ with a spatially periodic condition and a given

terminal condition at time $T > 0$:

$$
\begin{cases}
du(t,x) + \{\nu\Delta u(t,x) + (u\cdot\nabla)u(t,x) + (\sigma\cdot\nabla)Z(t,x) + \nabla p(t,x)\}\,dt \\
\quad = -f(t,x,u,Z)\,dt + Z(t,x)\,dW_t, \quad (t,x)\in[0,T)\times\mathbb{R}^2; \\
\qquad \operatorname{div} u(t,x) = 0, \quad (t,x)\in[0,T)\times\mathbb{R}^2; \\
\qquad u(t,x+ae_i) = u(t,x), \quad (t,x)\in[0,T)\times\mathbb{R}^2, \ \ i=1,2; \\
\qquad u(T,x) = \xi(x), \ \ (x,\omega)\in\mathbb{R}^2.
\end{cases}
\tag{1.3}
$$

Here σ is a measure of "correlation" between the Laplace and the Brownian motion, $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$, $a > 0$ is the period in the ith direction, $u = (u_1, u_2)$ is the random two-dimensional velocity field of a fluid in $\mathbb{R}^2$, f represents the external forces which allow for feedback involving the velocity field u and the stochastic process Z and may be inhomogeneous in time. The terminal status of the velocity field is a known random field ξ on the underlying probability space. For notational convenience, however, the variable $\omega \in \Omega$ in various functions and solutions will often be omitted.

It is worth noting that, though sharing the same name, our BSNSE essentially differs from that of Sundar and Yin [17] since the sign of the viscous term "$\nu\Delta u$" differs. Furthermore, we allow the external force f to depend on both unknown fields u and Z in a nonlinear way, and the drift term to depend on the gradient of the second unknown field Z.

In [9, 10], Cauchy problems for the (forward) stochastic Navier-Stokes equations in $\mathbb{R}^d$ driven by a random nonlinear force and a white noise are studied and the existence and uniqueness of a global martingale solution have been proved. As a motivation BSNSEs emerge in regard to inverse problems to determine the stochastic noise coefficients from the terminal velocity field as observed. In [6, 24], a stochastic representation in terms of Lagrangian paths for the backward incompressible Navier-Stokes equations without forcing is shown and used to prove the local existence of solutions in weighted Sobolev spaces and the global existence results in two dimensions or with a large viscosity. In [16, 17], the existence and uniqueness of adapted solutions are given to the backward stochastic Lorenz equations and to the backward stochastic Navier-Stokes equations (1.3) in a bounded domain with $\sigma \equiv 0$, $\nu < 0$ and the external force $f(t, y, z) \equiv f(t)$.

The rest of the paper is organized as follows. In Section 2, we introduce some notations, assumptions, and preliminary lemmas, and state the main result (see Theorem 2.1). In Section 3, we consider the spectral approximations and give relevant estimates. In Section 4, we prove the existence of an adapted solution to the projected finite dimensional systems for our 2D BSNSE. Finally, in Section 5, we give the proof of Theorem 2.1.

2 Preliminaries and the main results

Let $G = (0, a)^2$ be the rectangular of the period. For any nonnegative integer m, we denote by $H^m(G)$ the Sobolev space of functions which are in $L^2(G)$, together with all their derivatives of orders up to m and by $H^m_{pe}(G)$ the space of functions which belong to $H^m_{loc}(\mathbb{R}^2)$ (i.e., $u|_{\mathcal{O}} \in H^m(\mathcal{O})$ for every open bounded set $\mathcal{O}$) and which are periodic with period G. $H^m_{pe}(G)$ is a Hilbert space for the scalar product and the norm

$$(u, v)_m = \sum_{[\alpha]\leq m} \int_G D^\alpha u(x) D^\alpha v(x)\ dx, \quad |u|_m = \{(u, u)_m\}^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$, $[\alpha] = \alpha_1 + \alpha_2$, and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$$

with α_1 and α_2 being two nonnegative integers. The elements of $H^m_{pe}(G)$ are characterized by their Fourier series expansion:

$$H^m_{pe} = \{u :\ u = \sum_{k\in\mathbb{Z}^2} c_k e^{2i\pi k\cdot x/a},\ \bar{c_k} = c_{-k}, \sum_{k\in\mathbb{Z}^2} |k|^{2m} |c_k|^2 < \infty\}, \tag{2.1}$$

and the norm $|u|_m$ is equivalent to the norm $\left\{\sum_{k\in\mathbb{Z}^2}(1 + |k|^{2m})|c_k|^2\right\}^{1/2}$. Set a Hilbert subspace of $H^m_{pe}(G)$:

$$\dot{H}^m_{pe}(G) = \{u \in H^m_{pe}(G) :\ \text{in its Fourier expansion (2.1)},\ c_0 = 0\}, \tag{2.2}$$

with the norm $|u|_{m,0} = \{\sum_{k\in\mathbb{Z}^2} |k|^{2m}|c_k|^2\}^{1/2}$. Actually, through (2.1) and (2.2), we can define $H^m_{pe}(G)$ and $\dot{H}^m_{pe}(G)$ for arbitrary $m \in \mathbb{R}$. Moreover, $\dot{H}^m_{pe}(G)$ and $\dot{H}^{-m}_{pe}(G)$ are in duality for all $m \in \mathbb{R}$.

As in the framework of treating the deterministic Navier-Stokes equations (c.f. [15, 20, 21]), we set up three phase spaces of functions of the spatial variable $x \in G$ as follows:

$$\begin{aligned}
H &= \{\varphi \in \dot{H}^0_{pe}(G) \times \dot{H}^0_{pe}(G) : \operatorname{div}\varphi = 0 \text{ in } \mathbb{R}^2\},\\
V &= \{\varphi \in \dot{H}^1_{pe}(G) \times \dot{H}^1_{pe}(G) : \operatorname{div}\varphi = 0 \text{ in } \mathbb{R}^2\},\\
\mathscr{V} &= V \bigcap C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2),
\end{aligned}$$

where $C^\infty(\mathbb{R}^2)$ is the set of smooth functions on $\mathbb{R}^2$. Then both H and V are Hilbert spaces equipped with the respective scalar product and norm

$$\begin{aligned} \langle \phi,\ \varphi\rangle_H :&= \sum_{i=1}^{2}(\phi^i,\ \varphi^i)_0,\ \ \|\phi\|_H := \{\langle \phi,\ \phi\rangle_H\}^{1/2},\ \ \phi,\varphi \in H; \\ \langle \phi,\ \varphi\rangle_V :&= \sum_{i,j=1}^{2}(D_j^1\phi^i,\ D_j^1\varphi^i)_0,\ \ \|\phi\|_V := \{\langle \phi,\ \phi\rangle_V\}^{1/2},\ \ \phi,\varphi \in V. \end{aligned} \tag{2.3}$$

For simplicity, we denote $\|\cdot\|$ and $\langle\cdot,\ \cdot\rangle_H$ by $\|\cdot\|$ and $\langle\cdot,\ \cdot\rangle$ respectively. The dual product of $\psi\in V'$ and $\varphi \in V$ will be denoted by $\langle \psi,\varphi\rangle_{V',V}$ and it follows that

$$\langle \phi_1,\phi_2\rangle_{V',V} = \langle \phi_1,\phi_2\rangle \quad \phi_1\in V,\phi_2\in H.$$

For a little notational abuse we still denote by $\langle\cdot,\ \cdot\rangle$ the dual product $\langle\cdot,\ \cdot\rangle_{V',V}$. We shall use $|\cdot|$ to denote the absolute value or the Euclidean norm of $\mathbb{R}^2$. The set of all positive integers will be denoted by $\mathbb{Z}^+$ or $\mathbb{N}$. The Lebesgue measure of the domain G will be denoted by $|G|$. Define $\dot{\mathbb{H}}^m := \dot{H}^m_{pe}(G)\times \dot{H}^m_{pe}(G)$.

For any finite dimensional vector space F and $a,b\in F$, we denote by $a\cdot b$ the scalar product on F. Throughout this paper, we assume that the external force term $f(t,u,Z)$ and the terminal value term ξ are $\dot{\mathbb{H}}^0$-valued and $\dot{\mathbb{H}}^1$-valued respectively, so the solution pair (u,Z) of (1.3) must be $\dot{\mathbb{H}}^0\times\dot{\mathbb{H}}^0$-valued. By applying the projection $\mathbb{P}:\dot{\mathbb{H}}^0\to H$ (see [21]), since

$$H^\perp = \{\psi\in\dot{\mathbb{H}}^0 : \psi = \nabla q \ \text{ for some } \ q\in H^1_{pe}(G)\},$$

we can formulate the above terminal value problem of the 2D BSNSE (1.3) into the following problem to solve the backward stochastic evolutionary Navier-Stokes equation,

$$\left\{\begin{aligned} -du(t) =&\{-\nu Au(t) + B(u(t)) + JZ(t) + f(t,u(t),Z(t))\}dt \\ &- Z(t)dW_t, \quad t\in[0,T), \\ u(T) =&\xi, \end{aligned}\right. \tag{2.4}$$

where

$$\Pi(u,v) = \mathbb{P}((u\cdot\nabla)v) : V\times V\to V', \quad B(u) = \Pi(u,u) : V\to V',$$

$$JZ = \mathbb{P}\left((\sigma\cdot\nabla)Z\right), \quad \sigma(t,x) = (\sigma^1(t,x),\ \sigma^2(t,x)), \quad ((\sigma\cdot\nabla)Z)^i := \sum_{j=1}^{2}\sigma^j Z^i_{x_j},$$

and

$$A\varphi = \mathbb{P}(-\Delta\varphi) = -\Delta\varphi,$$

whose domain is $D(A) = \dot{\mathbb{H}}^2 \bigcap H$ and by Poincaré inequality we can show that $V = D(A^{1/2})$. Accordingly we shall adopt the equivalent norm $\|\varphi\|_V = \|\nabla\varphi\| = \|A^{1/2}\varphi\|$. Then all H, V, and $D(A)$ (with the graph norm $\|\cdot\|_{D(A)}$) are separable Hilbert spaces. With a little notational abuse we still use f and Z for the projections $\mathbb{P}(f)$ and $\mathbb{P}(Z)$, respectively,

Note that the above Stokes operator $A : D(A) \to H$ is positive definite, self-adjoint, and linear, and its resolvent is compact. Therefore, all the eigenvalues of A can be ordered into the increasing sequence $\{\lambda_i\}_{i=1}^{\infty}$. The corresponding eigenfunctions $\{e_i\}_{i=1}^{\infty}$ form a complete orthonormal basis for the space H, which is also a complete orthogonal (but not orthonormal) basis of the space V. With the identification $H = H'$ by the Riesz mapping, one has the triplet structure of compact (consequently continuous) embedding,

$$V \subset H \subset V'. \tag{2.5}$$

In what follows, $C > 0$ is a constant which may vary from line to line and we denote by $C(a_1, a_2, \cdots)$ or $C_{a_1,a_2,\cdots}$ a constant to depend on the parameters $a_1, a_2, \cdots$.

We make the following three assumptions.

Assumption (A1). The H-valued mapping f is defined on $\Omega \times [0,T] \times V \times H$ and for any $(u,z) \in V \times H$, $f(\cdot, u, z)$ is a predictable and H-valued process. Moreover, there exist a nonnegative constant β and a nonnegative adapted process $g \in L^{\infty}(\Omega, L^1([0,T]))$ such that the following conditions hold for all $v, v_1, v_2 \in V$, $\phi, \phi_1, \phi_2 \in H$ and $(\omega, t) \in \Omega \times [0,T]$:

(1). the map $s \mapsto \langle f(t, v_1 + sv_2, \phi),\ v\rangle$ is continuous on $\mathbb{R}$;

(2).

$$\begin{aligned}&\langle f(t, v_1, \phi_1) - f(t, v_2, \phi_2),\ v_1 - v_2\rangle \\ &\le \rho(v_2)\left(\|v_1 - v_2\|^2 + \|v_1 - v_2\|(\|\phi_1 - \phi_2\| + \|v_1 - v_2\|_V)\right);\end{aligned}$$

where $\rho : V \to (0, +\infty)$ is measurable and locally bounded;

(3).

$$\langle f(t, v, \phi),\ v\rangle \le g(t) + \epsilon\|\phi\|^2 + \varrho(\epsilon)\|v\|^2 + \beta\|v\|_V\|v\|,$$

where $\varrho : (0, 1] \to \mathbb{R}^+$ is continuous and decreasing;

(4).

$$\|f(t, v, \phi)\|^2 \le \left(g(t) + \beta(\|v\|_V^2 + \|\phi\|^2)\right)\rho_1(v),$$

where $\rho_1 : H \to (0, +\infty)$ is measurable and locally bounded.

Remark 1. In fact, (1) and (2) of Assumption (A1) implies $f(t,x,u,z)$ is locally Lipschitz continuous with respect to z in the following sense:

$$\|f(t,u,z)-f(t,u,Z)\|_{V'} \leq C(\|u\|_V)\|z-Z\|,$$

for all $(\omega,t)\in\Omega\times[0,T], u\in V$ and $z,Z\in H$. Actually, for any $\phi\in V-\{0\}, \epsilon\in\mathbb{R}^+$,

$$\langle f(t,u+\epsilon\phi,z)-f(t,u,Z),\ \phi\rangle \leq C(\|u\|_V)\|\phi\|_V\{\epsilon\|\phi\|+\|z-Z\|\}.$$

Letting $\epsilon\downarrow 0$, from the arbitrariness of ϕ we conclude that the local Lipschitz continuity holds.

Assumption (A2). The function σ^j defined on $\Omega\times[0,T]$ is real-valued $\mathcal{P}$-measurable such that $|\sigma^j|\leq\Lambda$, almost surely for $j=1,2$ and all $t\in[0,T]$, for some $\Lambda\in(0,\infty)$.

Assumption (A3) (super-parabolicity). There exist two constants $\lambda>0$ and $\bar{\lambda}>1$ such that

$$2\nu|\xi|^2-\bar{\lambda}^2\left(\sigma(t)\cdot\xi\right)^2\geq 2\lambda|\xi|^2$$

holds almost surely for all $(t,\xi)\in[0,T]\times\mathbb{R}^2$.

Note that, in Assumptions (A2) and (A3), our σ is defined independent of the spatial variable x.

For Banach space B and $p>1$, define

$$\mathscr{L}^p_{\mathscr{F}}(0,T;B) := \{\phi\in L^p(\Omega\times[0,T];B)\,|\,\{\phi(\cdot,t)\}_{0\leq t\leq T}\text{ is a predictable process}\}.$$

Define

$$M[0,T] := L^2_{\mathscr{F}}(\Omega;C([0,T];H))\cap\mathscr{L}^2_{\mathscr{F}}(0,T;V)$$

and

$$\mathscr{M} := \mathscr{M}[0,T] := M[0,T]\times\mathscr{L}^2_{\mathscr{F}}(0,T;H)$$

equipped with the norm

$$\|(u,Z)\|_{\mathscr{M}} = \left\{E\left[\sup_{t\in[0,T]}\|u(t)\|^2\right]+E\left[\int_0^T\|u(t)\|_V^2\,dt\right]+E\left[\int_0^T\|Z(t)\|^2dt\right]\right\}^{1/2}.$$

Throughout the paper, define

$$\Phi(t,\phi,\varphi) := -\nu A\phi+B(\phi)+J\varphi+f(t,\phi,\varphi),\ (\phi,\varphi)\in V\times H. \tag{2.6}$$

Definition 1. (weak solutions) For $\xi \in L^\infty_{\mathscr{F}_T}(\Omega; H)$ given, we say that $(u, Z) \in \mathscr{M}$ is a weak solution to (2.4) if for any $\varphi \in \mathscr{V}$, there holds almost surely

$$\begin{aligned}\langle u(t),\ \varphi\rangle = \langle \xi,\ \varphi\rangle + \int_t^T \langle \Phi(s, u(s), Z(s)),\ \varphi\rangle\ ds \\ - \int_t^T \langle Z(s),\ \varphi\rangle\ dW_s,\ \forall t \in [0, T].\end{aligned} \tag{2.7}$$

Definition 2. (strong solutions) For $\xi \in L^\infty_{\mathscr{F}_T}(\Omega; V)$ given, we say that (u, Z) is a strong solution to (2.4) if (u, Z) is a weak solution and

$$(u, Z) \in \left(L^2_{\mathscr{F}}(\Omega; C([0, T]; V)) \cap \mathscr{L}^2_{\mathscr{F}}(0, T; D(A))\right) \times \mathscr{L}^2_{\mathscr{F}}(0, T; V).$$

Remark 2. If we have verified that $(u, Z) \in (L^2_{\mathscr{F}}(\Omega; C([0, T]; V)) \cap \mathscr{L}^2_{\mathscr{F}}(0, T; D(A))) \times \mathscr{L}^2_{\mathscr{F}}(0, T; V)$ and that

$$u(t) = \xi + \int_t^T \Phi(s, u(s), Z(s))\ ds - \int_t^T Z(s)\ dW_s \quad a.s. \text{ in } H,$$

By the stochastic Fubini theorem (see, [12, Theorem 4.18]), we can check that (u, Z) is a strong solution to (2.4).

The main result of the paper is stated in the following theorem.

Theorem 2.1. *Under Assumptions* (A1)-(A3)*, for any* $\xi \in L^\infty_{\mathscr{F}_T}(\Omega; V)$*, the 2D BSNSE problem* (2.4) *admits a unique strong solution such that*

$$\begin{aligned}&\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[0,T]} \|u(s)\|_V^2 + E\left[\int_0^T \|u(s)\|^2_{D(A)} ds + \int_0^T \|Z(s)\|_V^2 ds\right] \\ &\le C\left\{\|g\|_{L^\infty(\Omega, L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathscr{F}_T}(\Omega;V)}\right\},\end{aligned} \tag{2.8}$$

where C *is a constant depending on* $\nu, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1$ *and* T*.*

For the trilinear mapping

$$b(u, v, w) := \langle \Pi(u, v),\ w\rangle = \sum_{i=1}^2 \sum_{j=1}^2 \int_G u_i \frac{\partial v_j}{\partial x_i} w_j\, dx,\ u, v, w \in H,$$

we have the following instrumental regularity properties.

Lemma 2.2. *The following properties hold for any two-dimensional bounded domain G, where C_G is used to denote different constants only depending on G.*

$$\begin{aligned}
&|b(u,v,w)| \le 2^{1/2}\|u\|_H^{1/2}\|u\|_V^{1/2}\|v\|_V\|w\|_H^{1/2}\|w\|_V^{1/2}, && u,v,w\in V,\\
&|b(u,v,w)| \le C_G\|u\|_H^{1/2}\|Au\|_H^{1/2}\|v\|_V\|w\|_H, && u\in D(A), v\in V, w\in H,\\
&|b(u,v,w)| \le C_G\|u\|_H^{1/2}\|u\|_V^{1/2}\|v\|_V^{1/2}\|Av\|_H^{1/2}\|w\|_H, && u\in V, v\in D(A), w\in H,\\
&|b(u,v,w)| \le C_G\|u\|_H\|v\|_V\|w\|_H^{1/2}\|Aw\|_H^{1/2}, && u\in H, v\in V, w\in D(A).
\end{aligned} \tag{2.9}$$

Moreover,

$$\begin{aligned}
&\langle \Pi(u,v),\ w\rangle = -\langle \Pi(u,w),\ v\rangle, \quad \text{for } u,v,w\in V,\\
&\langle \Pi(u,v),\ v\rangle = 0, \quad \text{for } u,v\in V.
\end{aligned} \tag{2.10}$$

For $u\in D(A)$, we have $B(u)\in H$,

$$\|B(u)\|_H \le C_G\|u\|_H^{1/2}\|u\|_V\|Au\|_H^{1/2}, \tag{2.11}$$

and especially, for the periodic case, we have

$$\langle \Pi(v,v),\ \Delta v\rangle = 0, \quad \text{for } v\in D(A) \ \textit{(c.f. [21, Lemma 3.1, Page 19])}.$$

The proof of Lemma 2.2 can be found in [19, 20]. The first inequality in (2.9) will be most useful in this work and its coefficient equals $2^{1/2}$ for any bounded and locally smooth domain in space dimension $n = 2$, which was proved in [19, Lemma 3.4]. The following lemma shows the regularity of functions in $H_0^1(G)$ for a 2D domain G, whose proof is available in [19].

Lemma 2.3. *For any two-dimensional open set G, we have*

$$\|v\|_{L^4(G)} \le 2^{1/4}\|v\|_{L^2(G)}^{1/2}\|\nabla v\|_{L^2(G)}^{1/2},\ v\in H_0^1(G). \tag{2.12}$$

We have the following two versions of Gronwall-Bellman inequalites, whose proof is referred to [2, 5].

(The Gronwall-Bellman Inequality): If a nonnegative scalar function $g(t)$ is continuous on $[0,T]$ and satisfies

$$g(t) \le (\ge) g(T) + \int_t^T (\alpha g(s) + h(s))\, ds, \quad t\in[0,T], \tag{2.13}$$

where $\alpha \geq 0$ is a constant and $h : [0, T] \to \mathbb{R}$ is integrable, then

$$g(t) \leq (\geq)\, e^{\alpha(T-t)} g(T) + \int_t^T e^{\alpha(s-t)} h(s)\ ds, \quad t \in [0, T]. \tag{2.14}$$

(The Stochastic Gronwall-Bellman Inequality): Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space whose filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ satisfies the usual conditions. Suppose $\{Y_s\}$ and $\{X_s\}$ are optional integrable processes and α is a nonnegative constant. If for all t, the map $s \mapsto E[Y_s|\mathcal{F}_t]$ is continuous almost surely and

$$Y_t \leq (\geq) E\left[\int_t^T (X_s + \alpha Y_s)\, ds + Y_T \;\Big|\; \mathcal{F}_t\right],$$

then we have almost surely

$$Y_t \leq (\geq) e^{\alpha(T-t)} E[Y_T|\mathcal{F}_t] + E\left[\int_t^T e^{\alpha(s-t)} X_s\, ds \;\Big|\; \mathcal{F}_t\right], \quad \forall t \in [0, T].$$

In this paper, we prove the existence and uniqueness of an adapted solution to the terminal value problem (1.3) of a two-dimensional backward stochastic Navier-Stokes equation with nonlinear forcing and the random perturbation driven by the Brownian motion. We use the spectral approximation, combined with the truncation and variational techniques, which is also a kind of compactness method. The methodology features an interactive analysis based on the regularity of the deterministic Navier-Stokes dynamics and the stochastic properties of the Itô-type diffusion processes.

3 Spectral approximations and estimates

In this section we consider the spectral approximation of the problem (2.4) obtained by orthogonally projecting the equation and the terminal data on the finite dimensional space

$$H_N = \mathrm{Span}\,\{e_1, e_2, \cdots, e_N\}.$$

Define

$$P_N : V' \to H_N, P_N f = \sum_{i=1}^N \langle f,\ e_i\rangle e_i, \quad f \in V'.$$

Then $\|P_N f\|^2 = \sum_{i=1}^N |\langle f, e_i\rangle|^2$ and P_N is the orthogonal projection on H_N, which is called the spectral projection. It is worth noting that $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent

in H_N and that $H_N = V_N := P_N V$. Define

$$\begin{aligned} A^N = P_N A,\ B^N(u) = P_N B(u),\ J^N Z = P_N JZ := \sum_{i=1}^{N} \langle JZ, e_i \rangle e_i; \\ f^N(\cdots) = P_N f(\cdots),\ Z^N(t) = P_N Z(t),\ \text{and } \xi^N = P_N \xi. \end{aligned} \tag{3.1}$$

Then the projected, N-dimensional problem of approximation to the problem (2.4) is defined to be

$$\begin{cases} du^N(t) = \big(\nu A^N u^N(t) - B^N(u^N(t)) - J^N Z^N(t) - f^N(t, u^N(t), Z^N(t))\big)\ dt \\ \qquad\qquad + Z^N(t)\, dW_t, \quad t \in [0, T); \\ u^N(T) = \xi^N. \end{cases} \tag{3.2}$$

Note that the projection does not affect the Brownian motion $\{W_t\}_{t \geq 0}$, and also that, the finite dimensional approximation equation (3.2) does not satisfy the conditions listed in [1].

We shall conduct *a priori* estimates for the adapted solution to the finite dimensional approximation problem (3.2).

First, by means of Young's inequality

$$ab \leq \frac{1}{p} a^p \varepsilon^p + \frac{1}{q\varepsilon^q} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad ab \geq 0,\ \varepsilon > 0,$$

under the Assumptions (A1)-(A3), we have

$$\begin{aligned} & 2\langle \Phi(t, \phi, \varphi),\ \phi \rangle - \|\varphi\|^2 \\ =\ & 2\langle -\nu A\phi + B(\phi) + J\varphi + f(t, \phi, \varphi),\ \phi \rangle - \|\varphi\|^2 \\ =& - 2\nu \langle A\phi,\ \phi \rangle - 2\langle f(t, \phi, \varphi),\ \phi \rangle - 2\langle \varphi,\ (\sigma \cdot \nabla)\phi \rangle - \|\varphi\|^2 \\ \leq & - 2\nu \|\phi\|_V^2 + 2(g(t) + \epsilon \|\varphi\|^2 + \varrho(\epsilon)\|\phi\|^2 + \beta \|\phi\|_V \|\phi\|) \\ & + 2\frac{1}{\bar{\lambda}} \|\varphi\| \|\bar{\lambda}(\sigma \cdot \nabla)\phi\| - \|\varphi\|^2 \quad \text{(choose } \epsilon \text{ small enough)} \\ \leq & - 2\lambda \|\phi\|_V^2 - \frac{\bar{\lambda}^2 - 1}{2\bar{\lambda}^2} \|\varphi\|^2 + 2g(t) + \frac{\bar{\lambda}^2 - 1}{4\bar{\lambda}^2} \|\varphi\|^2 + \lambda \|\phi\|_V^2 + C\|\phi\|^2 \\ =& - \lambda \|\phi\|_V^2 - \frac{\bar{\lambda}^2 - 1}{4\bar{\lambda}^2} \|\varphi\|^2 + 2g(t) + C\|\phi\|^2,\ (\phi, \varphi) \in V \times H, \end{aligned} \tag{3.3}$$

where the constant C depends only on $\lambda, \bar{\lambda}, \varrho$ and β.

Lemma 3.1. *Let the conditions of Theorem 2.1 hold. If $(u^N(\cdot), Z^N(\cdot)) \in \mathscr{M}$ is an adapted solution of the problem (3.2), then we have almost surely*

(1).

$$\sup_{t\in[0,T]} \left\{ \|u^N(t)\|^2 + E_{\mathcal{F}_t} \left[\int_t^T \|u^N(s)\|_V^2 + \|Z^N(s)\|^2 \, ds \right] \right\} \\ \leq C \left(\|g\|_{L^\infty(\Omega, L^1([0,T]))} + \|\xi\|^2_{L^\infty(\Omega,H)} \right), \tag{3.4}$$

where C is a constant depending only on $T, \nu, \lambda, \bar{\lambda}, \beta$ and ϱ;

(2).

$$\sup_{t\in[0,T]} \left\{ \|u^N(t)\|_V^2 + E_{\mathcal{F}_t} \left[\int_t^T \|Au^N(s)\|^2 + \|Z^N(s)\|_V^2 ds \right] \right\} \\ \leq C \left(\|g\|_{L^\infty(\Omega, L^1([0,T]))} + \|\xi\|^2_{L^\infty(\Omega,V)} \right) \tag{3.5}$$

with C being a constant depending only on $\nu, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1$ and T.

Proof. Applying the backward Itô formula to the scalar-valued, stochastic process $\|u^N(t)\|^2$, and noting that

$$\langle B^N(u^N(t)), u^N(t) \rangle = 0,$$

we have

$$\begin{aligned}
&\|u^N(t)\|^2 \\
&= \|\xi^N\|^2 - 2\nu \int_t^T \langle A^N u^N(s),\, u^N(s) \rangle \, ds + 2 \int_t^T \langle f^N(s, u^N(s), Z^N(s)),\, u^N(s) \rangle \, ds \\
&\quad + 2 \int_t^T \langle J^N Z^N(s),\, u^N(s) \rangle \, ds - 2 \int_t^T \langle Z^N(s),\, u^N(s) \rangle \, dW_s - \int_t^T \|Z^N(s)\|^2 \, ds \\
&= \|\xi^N\|^2 - 2\nu \int_t^T \langle A u^N(s),\, u^N(s) \rangle \, ds + 2 \int_t^T \langle f(s, u^N(s), Z^N(s)),\, u^N(s) \rangle \, ds \\
&\quad + 2 \int_t^T \langle J Z^N(s),\, u^N(s) \rangle \, ds - 2 \int_t^T \langle Z^N(s),\, u^N(s) \rangle \, dW_s - \int_t^T \|Z^N(s)\|^2 \, ds.
\end{aligned}$$

In view of (3.3), we have

$$
\begin{aligned}
&\|u^N(t)\|^2 \\
&= \|\xi^N\|^2 + \int_t^T \left(2\langle \Phi(s, u^N(s), Z^N(s)),\ u^N(s)\rangle - \|Z^N(s)\|^2\right) ds \\
&\quad - 2\int_t^T \langle Z^N(s),\ u^N(s)\rangle\, dW_s \\
&\le \|\xi^N\|^2 - 2\int_t^T \langle Z^N(s),\ u^N(s)\rangle\, dW_s \\
&\quad + \int_t^T \left(-\lambda \|u^N(s)\|_V^2 - \frac{\bar{\lambda}^2-1}{4\bar{\lambda}^2}\|Z^N(s)\|^2 + 2g(s) + C\|u^N(s)\|^2\right) ds
\end{aligned}
\tag{3.6}
$$

where the constant C is independent of N. Since

$$
\begin{aligned}
&E\left[\sup_{\tau\in[t,T]}\left|\int_\tau^T \langle Z^N(s),\ u^N(s)\rangle\ dW_s\right|\right] \\
&\le 2E\left[\sup_{\tau\in[t,T]}\left|\int_t^\tau \langle Z^N(s),\ u^N(s)\rangle\ dW_s\right|\right] \\
&\le CE\left[\left(\int_t^T \|Z^N(s)\|^2\|u^N(s)\|^2\ ds\right)^{1/2}\right] \quad \text{(by BDG inequality)} \\
&\le CE\left[\sup_{s\in[t,T]}\|u^N(s)\|\left(\int_t^T \|Z^N(s)\|^2\ ds\right)^{1/2}\right] \\
&\le (1/2)E\left[\sup_{s\in[t,T]}\|u^N(s)\|^2\right] + CE\left[\int_t^T \|Z^N(s)\|^2\ ds\right],
\end{aligned}
\tag{3.7}
$$

taking the conditional expectation on both sides of the second inequality of (3.6), we obtain

$$
\begin{aligned}
&\|u^N(t)\|^2 + \lambda E_{\mathscr{F}_t}\left[\int_t^T \|u^N(s)\|_V^2\ ds\right] + \frac{\bar{\lambda}^2-1}{4\bar{\lambda}^2}E_{\mathscr{F}_t}\left[\int_t^T \|Z^N(s)\|^2\ ds\right] \\
&\le E_{\mathscr{F}_t}\left[\|\xi\|^2\right] + CE_{\mathscr{F}_t}\left[\int_t^T \left(g(s) + \|u^N(s)\|^2\right)\ ds\right],\ a.s..
\end{aligned}
$$

From the stochastic Gronwall-Bellman inequality, it follows that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left\{\|u^N(t)\|^2 + E_{\mathcal{F}_t}\left[\int_t^T \|u^N(s)\|_V^2\ ds\right] + E_{\mathcal{F}_t}\left[\int_t^T \|Z^N(s)\|^2\ ds\right]\right\} \\
&\le C\left(\|g\|_{L^\infty(\Omega,L^1([0,T]))} + \|\xi^N\|^2_{L^\infty(\Omega,H)}\right) \\
&\le C\left(\|g\|_{L^\infty(\Omega,L^1([0,T]))} + \|\xi\|^2_{L^\infty(\Omega,H)}\right),\ a.s.
\end{aligned}
\tag{3.8}
$$

where C is a constant depending only on $T, \nu, \lambda, \bar{\lambda}, \beta$ and ϱ.

On the other hand, as $(B(u),\ \Delta u) = 0$, using Itô formula, we have

$$
\begin{aligned}
\|u^N(t)\|_V^2 = \|\xi^N\|_V^2 &- 2\nu\int_t^T \langle A^N u^N(s),\ A^N u^N(s)\rangle\, ds \\
&+ 2\int_t^T \langle f^N(s, u^N(s), Z^N(s)),\ A^N u^N(s)\rangle\, ds \\
&+ 2\int_t^T \langle J^N Z^N(s),\ A^N u^N(s)\rangle\, ds \\
&- 2\int_t^T \langle Z^N(s),\ A^N u^N(s)\rangle\, dW_s - \int_t^T \|Z^N(s)\|_V^2\, ds \\
= \|\xi^N\|_V^2 &- 2\nu\int_t^T \|Au^N(s)\|^2\, ds + 2\int_t^T \langle f(s, u^N(s), Z^N(s)),\ Au^N(s)\rangle\, ds \\
&- \int_t^T \|Z^N(s)\|_V^2\, ds - 2\int_t^T \sum_{i=1}^2 \langle \nabla (Z^N)^i(s),\ (\sigma\cdot\nabla)\nabla(u^N)^i(s))\rangle\, ds \\
&- 2\int_t^T \langle Z^N(s),\ Au^N(s)\rangle\, dW_s,\ t\in[0,T].
\end{aligned}
$$

By the integration-by-parts formula and the fact that the integrals on the boundary ∂G of G vanish, we obtain

$$
\|A\phi\|^2 = \sum_{i=1}^2 \|\nabla\phi^i\|_V^2, \quad \forall\, \phi\in D(A).
$$

It follows that

$$
\begin{aligned}
&-2\nu\int_t^T\|Au^N(s)\|^2\,ds-2\int_t^T\sum_{i=1}^2\langle\nabla(Z^N)^i(s),\ (\sigma\cdot\nabla)\nabla(u^N)^i(s))\rangle\,ds\\
\le\ &-2\nu\sum_{i=1}^2\int_t^T\|\nabla(u^N(s))^i\|_V^2\,ds+\int_t^T\sum_{i=1}^2\|\bar\lambda(\sigma\cdot\nabla)\nabla(u^N)^i(s)\|^2\,ds\\
&+\bar\lambda^{-2}\int_t^T\|Z^N(s)\|_V^2\,ds\\
\le\ &-2\lambda\sum_{i=1}^2\int_t^T\|\nabla(u^N(s))^i\|_V^2\,ds+\bar\lambda^{-2}\int_t^T\|Z^N(s)\|_V^2\,ds,\ a.s.,\forall\, t\in[0,T].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\|u^N(t)\|_V^2\\
\le\ &\|\xi\|_V^2-2\int_t^T\langle Z^N(s),\ Au^N(s)\rangle\,dW_s-\frac{\bar\lambda^2-1}{2\bar\lambda^2}\int_t^T\|Z^N(s)\|_V^2\,ds\\
&-2\lambda\int_t^T\|Au^N(s)\|^2\,ds+2\int_t^T\langle f(s,u^N(s),Z^N(s)),\ Au^N(s)\rangle\,ds\\
\le\ &\|\xi\|_V^2-2\int_t^T\langle Z^N(s),\ Au^N(s)\rangle\,dW_s-\frac{\bar\lambda^2-1}{2\bar\lambda^2}\int_t^T\|Z^N(s)\|_V^2\,ds\\
&-\lambda\int_t^T\|Au^N(s)\|^2ds+\frac{1}{\lambda}\int_t^T\|f(s,u^N(s),Z^N(s))\|^2\,ds\\
\le\ &\|\xi\|_V^2-2\int_t^T\langle Z^N(s),\ Au^N(s)\rangle\,dW_s-\frac{\bar\lambda^2-1}{2\bar\lambda^2}\int_t^T\|Z^N(s)\|_V^2\,ds\\
&-\lambda\int_t^T\|Au^N(s)\|^2ds+\frac{1}{\lambda}\int_t^T\left[g(s)+\beta(\|u^N(s)\|_V^2+\|Z^N(s)\|^2)\right]\rho_1(u^N)\,ds\\
\le\ &\|\xi\|_V^2-2\int_t^T\langle Z^N(s),\ Au^N(s)\rangle\,dW_s-\frac{\bar\lambda^2-1}{2\bar\lambda^2}\int_t^T\|Z^N(s)\|_V^2\,ds\\
&-\lambda\int_t^T\|Au^N(s)\|^2\,ds+C\int_t^T\left[g(s)+\beta(\|u^N(s)\|_V^2+\|Z^N(s)\|^2)\right]\,ds.
\end{aligned}
\tag{3.9}
$$

As

$$
\begin{aligned}
&E\left[\sup_{\tau\in[t,T]}\left|\int_\tau^T \langle Z^N(s), Au^N(s)\rangle dW_s\right|\right]\\
&\leq 2E\left[\sup_{\tau\in[t,T]}\left|\sum_{i=1}^2\int_t^\tau \langle \nabla(Z^N)^i(s), \nabla(u^N)^i(s)\rangle dW_s\right|\right]\\
&\leq CE\left[\left(\int_t^T \|Z^N(s)\|_V^2\|u^N(s)\|_V^2 ds\right)^{1/2}\right] \qquad \text{(by BDG inequality)} \qquad (3.10)\\
&\leq (1/2)E\left[\sup_{s\in[t,T]}\|u^N(s)\|_V^2\right] + CE\left[\int_t^T \|Z^N(s)\|_V^2 ds\right]\\
&\leq C(N)\left\{E\left[\sup_{s\in[t,T]}\|u^N(s)\|_V^2\right] + E\left[\int_t^T \|Z^N(s)\|_V^2 ds\right]\right\},
\end{aligned}
$$

taking conditional expectation on both sides of (3.9), we get

$$
\begin{aligned}
&\|u^N(t)\|_V^2 + \lambda E_{\mathscr{F}_t}\left[\int_t^T \|Au^N(s)\|^2\, ds\right] + \frac{\bar\lambda^2-1}{2\bar\lambda^2}E_{\mathscr{F}_t}\left[\int_t^T \|Z^N(s)\|_V^2\, ds\right]\\
&\leq E_{\mathscr{F}_t}\left[\|\xi\|_V^2\right] + CE_{\mathscr{F}_t}\left[\int_t^T (g(s) + \|u^N(s)\|_V^2 + \|Z^N(s)\|^2)\, ds\right].
\end{aligned} \tag{3.11}
$$

In view of (3.8), we conclude that, with probability 1,

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left\{\|u^N(t)\|_V^2 + E_{\mathcal{F}_t}\left[\int_t^T \|Au^N(s)\|^2 ds\right] + E_{\mathcal{F}_t}\left[\int_t^T \|Z^N(s)\|_V^2 ds\right]\right\}\\
&\leq C\left(\|g\|_{L^\infty(\Omega,L^1([0,T]))} + \|\xi\|^2_{L^\infty(\Omega,V)}\right),
\end{aligned} \tag{3.12}
$$

where C is a constant depending only on $\nu, \lambda, \bar\lambda, \beta, \varrho, \rho_1$ and T. $\square$

Lemma 3.2. *For any $u, v \in V$ and $\phi, \varphi \in H$,*

$$
|\langle B(u) - B(v), u - v\rangle| \leq \frac{\lambda}{4}\|u-v\|_V^2 + \frac{2}{\lambda}\|v\|_V^2\|u-v\|^2. \tag{3.13}
$$

Moreover, under Assumptions (A1)-(A3), there exists a positive constant K depending on $\bar\lambda$ such that

$$
\begin{aligned}
&-2\langle \Phi(t,u,\phi) - \Phi(t,v,\varphi), w\rangle\\
&+\|w\|^2\left(K + \frac{4}{\lambda}\|v\|_V^2 + K\rho^2(v)\right) + \frac{\bar\lambda^2+1}{2\bar\lambda^2}\|\bar w\|^2 \geq \frac{\lambda}{2}\|w\|_V^2,
\end{aligned} \tag{3.14}
$$

holds almost surely for any $t \in [0,T]$, $u, v \in V$ and $\varphi, \phi \in H$ with $w := u - v, \bar{w} := \phi - \varphi$, and Φ being defined by (2.6). Define

$$r_1(t) = \int_0^t \left(K + \frac{4}{\lambda} \|u(s)\|_V^2 + K\rho^2(u(s)) \right) ds$$

and

$$r_2(t) = \int_0^t \left(K + \frac{4}{\lambda} \|v(s)\|_V^2 + K\rho^2(v(s)) \right) ds,$$

for arbitrary $u, v \in L^2_{\mathscr{F}}(\Omega; L^2(0,T;V))$, and let $w(\cdot) = u(\cdot) - v(\cdot)$. Then for any $\phi, \varphi \in L^2_{\mathscr{F}}(\Omega; L^2(0,T;H))$ and $\bar{w}(\cdot) := \phi(\cdot) - \varphi(\cdot)$, we have for $i = 1, 2$

$$-\langle 2\Phi(t,u,\phi) - 2\Phi(t,v,\varphi) + \frac{dr_i(t)}{dt} w, w \rangle + \frac{1+\bar{\lambda}^2}{2\bar{\lambda}^2} \|\bar{w}\|^2 \geq 0, \ a.s.. \tag{3.15}$$

Proof. Let $w = u - v$. Then

$$\begin{aligned} & \langle B(u) - B(v), u - v \rangle \\ = & -\langle \Pi(u,w), u \rangle + \langle \Pi(v,w), v \rangle \\ = & -\langle \Pi(u,w), v \rangle + \langle \Pi(v,w), v \rangle = -\langle B(w), v \rangle. \end{aligned}$$

By the first inequality in Lemma 2.2 we can get

$$\begin{aligned} & |\langle B(u) - B(v), u - v \rangle_{V',V}| = |\langle B(w), v \rangle| = |\langle \Pi(w,v), w \rangle_{V',V}| \\ & \leq 2^{1/2} \|u - v\| \|u - v\|_V \|v\|_V \leq \frac{\lambda}{4} \|u - v\|_V^2 + \frac{2}{\lambda} \|u - v\|^2 \|v\|_V^2. \end{aligned}$$

It follows from the Assumptions (A1)-(A3) that

$$\begin{aligned} & -2\nu \langle Aw, w \rangle + 2\langle J\bar{w}, w \rangle + 2\langle f(t,u,\phi) - f(t,v,\varphi), w \rangle \\ \leq & -2\nu \|w\|_V^2 + 2\langle \bar{w}, \ (\sigma \cdot \nabla) w \rangle + 2\rho(v) \|w\|^2 + 2\rho(v) \|w\| (\|w\|_V + \|\bar{w}\|) \\ \leq & -\frac{3\lambda}{2} \|w\|_V^2 + \frac{1}{\bar{\lambda}^2} \|\bar{w}\|^2 + (1 - \frac{1+\bar{\lambda}^2}{2\bar{\lambda}^2}) \|\bar{w}\|^2 \\ & + (C(\bar{\lambda})\rho^2(v) + 2\rho(v)) \|w\|^2 \\ \leq & -\frac{3\lambda}{2} \|w\|_V^2 + \frac{1+\bar{\lambda}^2}{2\bar{\lambda}^2} \|\bar{w}\|^2 + (K + K\rho^2(v)) \|w\|^2, \end{aligned} \tag{3.16}$$

where the constant K only depends on $\bar{\lambda}$. Hence, in view of (3.13), we obtain (3.14).

Then (3.15) for $i = 2$ follows from (3.14) by direct calculation. The case of $i = 1$ in (3.15) is shown in a similar way. $\square$

4 Solutions of the finite dimensional systems

In this section, we consider the existence of an adapted solution to the projected, N-dimensional problem (3.2) of the 2D backward stochastic Navier-Stokes equations which we also call the finite dimensional system. To solve the finite dimensional system (3.2), we shall make use of the result of Briand et al. [1].

Consider the following backward stochastic differential equation (BSDE in short):

$$Y(t) = \zeta + \int_t^T g(s, Y(s), q(s))\, ds - \int_t^T q(s)\, dW_s, \tag{4.1}$$

where ζ is an $\mathbb{R}^N$-valued $\mathscr{F}_T$-measuable random vector and the random function

$$g: \ [0,T] \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$$

is $\mathcal{P} \times \mathscr{B}(\mathbb{R}^N) \times \mathscr{B}(\mathbb{R}^N)$-measurable.

The following lemma comes from [1, Theorem 4.2].

Lemma 4.1. *Assume that g and ζ satisfy the following four conditions.*

(C1). For some $p > 1$, we have

$$E\left[|\zeta|^p + \left(\int_0^T |g(t,0,0)|\, ds \right)^p \right] < \infty.$$

(C2). There exist constants $\alpha \geq 0$ and $\mu \in \mathbb{R}$ such that almost surely we have for each $(t, y, y', z, z') \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$,

$$|g((t,y,z) - g(t,y,z')| \leq \alpha |z - z'|, \tag{4.2}$$

$$\langle y - y', g(t,y,z) - g(t,y',z) \rangle \leq \mu |y - y'|^2 \qquad \text{(monotonicity condition)}. \tag{4.3}$$

(C3). The function $y \mapsto g(t,y,z)$ is continuous for any $(t,z) \in [0,T] \times \mathbb{R}^N$.

(C4). For any $r > 0$, the random process

$$\left\{ \psi_r(t) := \sup_{|y| \leq r} |g(t,y,0) - g(t,0,0)|,\ t \in [0,T] \right\}$$

lies in the space $L^1(\Omega \times [0,T])$. Then BSDE (4.1) *admits a unique solution $(Y, q) \in L^p(\Omega, C([0,T], \mathbb{R}^N)) \times Ł^p(\Omega, L^2([0,T], \mathbb{R}^N))$.*

Remark 3. It is worth noting that our finite dimensional system does not satisfy the monotonicity condition (C2). In fact, by Lemma 3.2 our finite dimensional system only satisfies a local monotonicity condition in some sense, which prevents us to directly use this lemma to our finite dimensional system.

Lemma 4.2. *For any $M, N \in \mathbb{Z}^+$, define the function of truncation $R_M(\cdot)$ to be a C^2 function on H_N such that for $X = \sum_{i=1}^N x_i e_i$,*

$$R_M(X) = \begin{cases} 1, & \text{if } \|X\| \le M; \\ \in (0,1), & \text{if } M < \|X\| < M+1; \\ 0, & \text{if } \|X\| \ge M+1. \end{cases}$$

Thus $R_M(\cdot)$ is uniformly Lipschitz continuous. For each $n \in \mathbb{Z}^+$, denote $\varphi_n(z) = zn/(\|z\| \vee n)$, $z \in H^N$ and set

$$\Phi^{N,M,n}(t,y,z) = R_M(y) \frac{n}{h_M(t) \vee n} P_N \Phi(t, y, \varphi_n(z)),$$

where

$$\begin{aligned} h_M(t) &= 4 \left\{ \left(g(t) + \beta C_N (M+1)^2\right) \operatorname*{ess\,sup}_{\|w\| \le M+1} |\rho_1(w)| + C_{N,\nu}(M+1)^2 \right\}^{1/2} \\ &\ge \operatorname*{ess\,sup}_{\|w\| \le M+1} \|\Phi(t,w,0)\| \end{aligned} \tag{4.4}$$

and $h_M \in L^1(\Omega \times [0,T])$. Then under Assumptions (A1)-(A3), $\Phi^{N,M,n}$ satisfies the conditions (C2)-(C4) in Lemma 4.1.

Proof. Under Assumptions (A1)-(A3) and Remark 1, we only need verify (4.3), i.e., there is a uniform constant $C_{N,M,n} > 0$ such that

$$\langle \Phi^{N,M,n}(t,X,Z) - \Phi^{N,M,n}(t,Y,Z),\ X - Y \rangle \le C_{N,M,n} \|X - Y\|^2, \quad a.s., \tag{4.5}$$

for any $X, Y, Z \in H_N$ and all $t \in [0,T]$. For any $X, Y \in H_N$, inequality (4.5) holds trivially if $\|X\| > M+1$ and $\|Y\| > M+1$. Thus, it is sufficient to consider the

case of $\|Y\| \leq M+1$. We have

$$
\begin{aligned}
& \langle \Phi^{N,M,n}(t,X,Z) - \Phi^{N,M,n}(t,Y,Z),\ X-Y\rangle \\
= {} & R_M(X)\frac{n}{h_M(t)\vee n}\langle \Phi(t,X,\varphi_n(Z)) - \Phi(t,Y,\varphi_n(Z)),\ X-Y\rangle \\
& + \frac{n}{h_M(t)\vee n}(R_M(X)-R_M(Y))\langle \Phi^{N,M,n}(t,Y,\varphi_n(Z)),\ X-Y\rangle \\
& \qquad \text{(by (3.14) of Lemma 3.2)} \\
\leq {} & \left(K + \frac{4}{\lambda}\|Y\|^2 + K\rho^2(Y)\right)\|X-Y\|^2 \\
& \quad + C_M\|X-Y\|^2\frac{n}{h_M(t)\vee n}\|\Phi^{N,M,n}(t,Y,\varphi_n(Z))\| \\
\leq {} & \left(K + \frac{4}{\lambda}\|Y\|^2 + K\rho^2(Y)\right)\|X-Y\|^2 \\
& \quad + C_M\|X-Y\|^2\frac{n}{h_M(t)\vee n}(h_M(t) + C_{N,M}\cdot n) \\
\leq {} & C_{M,N,n}\|X-Y\|^2,
\end{aligned}
\tag{4.6}
$$

which completes the proof. □

Theorem 4.3. *Let Assumptions* (A1)-(A3) *hold. For any $\xi \in L^\infty_{\mathscr{F}}(\Omega; V)$, the projected problem* (3.2) *admits a unique adapted solution $(u^N(\cdot), Z^N(\cdot)) \in \mathscr{M}$ for each given positive integer N, which satisfies*

$$
\|(u^N, Z^N)\|_{\mathscr{M}} \leq C\left\{1 + E\left[\|\xi\|^2\right]\right\}, \tag{4.7}
$$

where C is a constant independent of N.

Proof. **Step 1**. Let us verify the uniqueness part. Suppose (u^N, Z^N) and (v^N, Y^N) are two solutions of the projected problem (3.2). Note that the a priori estimates in Lemma 3.1 holds for both (u^N, Z^N) and (v^N, Y^N). Denote by (U^N, X^N) the pair of processes $(u^N - v^N, Z^N - Y^N)$. Define

$$
r(t) := \int_0^t \left[K + \frac{4}{\lambda}\|v^N(s)\|_V^2 + K\rho^2(v^N(s))\right] ds.
$$

An application of Itô formula and Lemma 3.2 yields

$$
\begin{aligned}
&e^{r(t)}\|U^N(t)\|^2 \\
&= \int_t^T e^{r(s)} \Big[2 \left\langle \Phi(s, u^N(s), Z^N(s)) - \Phi(s, v^N(s), Y^N(s)),\, U^N(s) \right\rangle \\
&\qquad - \|X^N\|^2 - \|U^N(s)\|^2 \left(K + \frac{4}{\lambda}\|v^N(s)\|_V^2 + K\rho^2(v^N(s)) \right) \Big]\, ds \\
&\quad - 2\int_t^T e^{r(s)} \langle U^N(s), X^N(s)\rangle dW_s \\
&\le -\frac{\bar\lambda^2 - 1}{2\bar\lambda^2} \int_t^T e^{r(s)} \|X^N(s)\|^2 ds - 2\int_t^T e^{r(s)} \langle U^N(s), X^N(s)\rangle\, dW_s.
\end{aligned} \tag{4.8}
$$

Taking conditional expectations on both sides, we get

$$
e^{r(t)}\|U^N(t)\|^2 + \frac{\bar\lambda^2 - 1}{2\bar\lambda^2} E_{\mathcal{F}_t}\left[\int_t^T e^{r(s)} \|X^N(s)\|^2 ds \right] \le 0, \quad a.s., \text{ for any } t \in [0,T],
$$

which implies the uniqueness.

Step 2. For any $N, M, n \in \mathbb{Z}^+$, following Lemma 4.2, we can verify that the pair $(\xi^N, \Phi^{N,M,n})$ satisfies the conditions (C1)-(C4) in Lemma 4.1. Hence, by Lemma 4.1 there exists a unique solution $(u^{N,M,n}, Z^{N,M,n}) \in \mathscr{M}$ to the following BSDE:

$$
u^{N,M,n}(t) = \xi^N + \int_t^T \Phi^{N,M,n}(s, u^{N,M,n}(s), Z^{N,M,n}(s))\, ds - \int_t^T Z^{N,M,n}(s)\, dW_s. \tag{4.9}
$$

In a similar way to Lemma 3.1, we deduce that there exists a positive constant K_1 which is independent of N, M and n such that

$$
\sup_{t\in[0,T]} \|u^{N,M,n}(t)\| + E\left[\int_0^T \|Z^{N,M,n}(s)\|^2\, ds \right] \le K_1, \quad a.s.. \tag{4.10}
$$

Therefore, taking $M > K_1$, we have $R_M(u^{N,M,n}(s)) \equiv 1$ and $(u^{N,M,n}, Z^{N,M,n})$ is independent of M. Thus, we write $(u^{N,n}, Z^{N,n})$ instead of $(u^{N,M,n}, Z^{N,M,n})$ below. Moreover, there exists a positive constant K_2 independent of n such that

$$
\begin{aligned}
&K + \frac{4}{\lambda}\|u^{N,n}(t)\|_V^2 + K\rho^2(u^{N,n}(t)) \le K_2, \\
&\|\Phi(t, u^{N,n}(t), \phi_1) - \Phi(t, u^{N,n}(t), \phi_2)\| \le K_2\|\phi_1 - \phi_2\|, \quad dP \otimes dt\text{-almost},
\end{aligned} \tag{4.11}
$$

holds for all $\phi_1, \phi_2 \in H$ and $N, n \in \mathbb{Z}^+$.

For $j \in \mathbb{Z}^+$, set $(U^N, X^N) = (u^{N,n+j} - u^{N,n}, Z^{N,n+j} - Z^{N,n})$. Applying Itô formula similar to (4.8), we get

$$
\begin{aligned}
& e^{K_2 t}\|U^N(t)\|^2 + \frac{\bar{\lambda}^2 - 1}{2\bar{\lambda}^2}\int_t^T e^{K_2 s}\|X^N(s)\|^2\, ds \\
\le\ & 2\int_t^T e^{K_2 s}\langle \Phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \Phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s)),\ U^N(s)\rangle ds \\
& - 2\int_t^T e^{K_2 s}\langle U^N(s), X^N(s)\rangle\, dW_s \qquad (4.12) \\
& \text{(by (4.10))} \\
\le\ & 4K_1\int_t^T e^{K_2 s}\|\Phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \Phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s))\|\, ds \\
& - 2\int_t^T e^{K_2 s}\langle U^N(s), X^N(s)\rangle\, dW_s.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& E\left[\sup_{\tau\in[t,T]} \left|\int_\tau^T e^{K_2 s}\langle U^N(s), X^N(s)\rangle\, dW_s\right|\right] \\
\le\ & CE\left[\left(\int_t^T e^{2K_2 s}\|X^N(s)\|^2\|U^N(s)\|^2\, ds\right)^{1/2}\right] \text{(by BDG inequality)} \qquad (4.13) \\
\le\ & \epsilon E\left[\sup_{s\in[t,T]}(e^{K_2 s}\|U^N(s)\|^2)\right] + C_\epsilon E\left[\int_t^T \|X^N(s)\|^2 e^{K_2 s}\, ds\right],
\end{aligned}
$$

with the positive constant ϵ to be determined later. Then choosing ϵ to be small enough, we deduce from (4.12) that

$$
\begin{aligned}
& \|(U^N, X^N)\|_{\mathscr{M}} \\
\le\ & CE\left[\int_0^T \|\Phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \Phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s))\|\, ds\right].
\end{aligned}
$$

As

$$
\begin{aligned}
& \|\Phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \Phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s))\| \\
\le\ & 2K_2\|Z^{N,n}(s)\|\mathbb{I}_{\{\|Z^{N,n}(s)\|>n\}} + 2K_2\|Z^{N,n}(s)\|\mathbb{I}_{\{h_{K_1}(s)>n\}} + 2h_{K_1}(s)\mathbb{I}_{\{h_{K_1}(s)>n\}},
\end{aligned} \qquad (4.14)
$$

in view of (4.10) and $h_{K_1} \in L^1(\Omega\times[0,T])$, we conclude that $(u^{N,n}, Z^{N,n})$ is a Cauchy sequence in $\mathscr{M}$. Denote the limit by $(u^N, Z^N) \in \mathscr{M}$. It is easily checked that (u^N, Z^N) is a solution of the projected problem (3.2).

Step 3. Estimate (4.7) follows from Lemma 3.1, which completes the proof.

$\square$

5 Proof of Theorem 2.1

Proof of Theorem 2.1. Our proof consists of the following four steps.

Step 1. By Theorem 4.3, we have solved the projected problem (3.2) in $\mathscr{M}$. By Lemma 3.1, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[0,T]} \|u^N(s)\|_V^2 + E\left[\int_0^T \|Au^N(s)\|^2 + \|Z^N(s)\|_V^2 ds\right] \\ &\leq C\left(\|g\|_{L^\infty(\Omega,L^1([0,T]))} + \|\xi\|^2_{L^\infty(\Omega,V)}\right),\end{aligned} \tag{5.1}$$

where C is a constant depending only on $\nu, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1$ and T . Since we get $\|B(v)\|^2 \leq C_G\|v\|\|v\|_V^2\|Av\|$ from Lemma 2.2, under Assumptions (A1)-(A3), we conclude

$$E\left[\int_0^T \left(\|B(u^N(s))\|^2 + \|(\sigma\cdot\nabla)Z^N(s)\|^2 + \|f(s,u^N(s),Z^N(s))\|^2\right) ds\right] \leq C.$$

Hence,

$$\|P_N\Phi(\cdot,u^N,Z^N)\|_{L^2(\Omega\times[0,T],H)} \leq \|\Phi(\cdot,u^N,Z^N)\|_{L^2(\Omega\times[0,T],H)}\| \leq C. \tag{5.2}$$

All the constants Cs above are independent of N.

Step 2. Now we consider the weak convergence. Clearly,

$$\xi^N \to \xi \text{ strongly in } V, a.s., \text{ and } \|\xi^N\|_V \leq \|\xi\|_V \text{ as } N\to\infty,$$

which implies that $\xi^N \to \xi$ in $L^p(\Omega, V)$ for any $p \in (1, +\infty)$. Then the following weak and weak star convergence results in respective spaces hold: there exists a subsequence $\{N_k\}_{k=1}^\infty$ of $\{N\}$, such that, as $k\to\infty$,

$$\begin{aligned}u^{N_k}(\cdot) &\xrightarrow{w} u(\cdot) \text{ in } \mathscr{L}^2_{\mathscr{F}}(0,T;D(A))),\\ u^{N_k}(\cdot) &\xrightarrow{w^*} u(\cdot) \text{ in } L^\infty_{\mathscr{F}}(\Omega;C([0,T];V)),\\ Z^{N_k}(\cdot) &\xrightarrow{w} Z(\cdot) \text{ in } L^2_{\mathscr{F}}(\Omega;L^2(0,T;V)),\\ \Phi(\cdot,u^{N_k},Z^{N_k})(\cdot) &\xrightarrow{w} \Gamma(\cdot) \text{ in } L^2_{\mathscr{F}}(\Omega;L^2(0,T;H)),\\ P_N\Phi(\cdot,u^{N_k},Z^{N_k})(\cdot) &\xrightarrow{w} \Psi(\cdot) \text{ in } L^2_{\mathscr{F}}(\Omega;L^2(0,T;H)),\end{aligned} \tag{5.3}$$

where u, Z, Γ and Ψ are some functions in the respective spaces.

By the Burkhölder-Davis-Gundy (BDG in short) inequality, we can get

$$
\begin{aligned}
E\left[\int_0^T \left\|\int_t^T Z^{N_k}(s)dW_s\right\|_V^2 dt\right] &\le TE\left[\sup_{t\in[0,T]}\left\|\int_t^T Z^{N_k}(s)dW_s\right\|_V^2\right] \\
&\le 2TE\left[\sup_{t\in[0,T]}\left\|\int_0^t Z^{N_k}(s)dW_s\right\|_V^2\right] + 2TE\left[\left\|\int_0^T Z^{N_k}(s)dW_s\right\|_V^2\right] \\
&\le 4TE\left[\sup_{t\in[0,T]}\left\|\int_0^t Z^{N_k}(s)dW_s\right\|_V^2\right] \le 4L_1TE\left[\int_0^T \|Z^{N_k}(s)|_V^2 ds\right]
\end{aligned}
\tag{5.4}
$$

where $L_1 > 0$ is a uniform constant from the BDG inequality. Hence, as a bounded linear operator on the space $L^2_{\mathscr{F}}(\Omega; L^2(0,T;V))$, the mapping

$$
\Upsilon : Z^{N_k}(\cdot) \longmapsto \int_\cdot^T Z^{N_k}(s)\, dW_s
$$

maps the weakly convergent sequence $\{Z^{N_k}(\cdot)\}$ to a weakly convergent sequence $\left\{\int_\cdot^T Z^{N_k}(s)dW_s\right\}$ in $L^2_{\mathscr{F}}(\Omega; L^2(0,T;V))$ such that

$$
\int_\cdot^T Z^{N_k}(s)\, dW_s \xrightarrow{w} \int_\cdot^T Z(s)\, dW_s \text{ in } L^2_{\mathscr{F}}(\Omega; L^2(0,T;V)), \quad \text{as } k \to \infty. \tag{5.5}
$$

Similarly it can be shown that, as $k \to \infty$,

$$
\int_\cdot^T P_{N_k}\Phi(s, u^{N_k}(s), Z^{N_k}(s))\, ds \xrightarrow{w} \int_\cdot^T \Psi(s)\, ds \text{ in } L^2_{\mathscr{F}}(\Omega; L^2(0,T;H)). \tag{5.6}
$$

Define

$$
\bar{u}(t) = \xi + \int_t^T \Psi(s)\, ds - \int_t^T Z(s)\, dW_s. \tag{5.7}
$$

It is easily checked that $\bar{u} = u$, $P \otimes dt$-almost. In view of [13, Theorem 4.2.5], we conclude that $u \in L^\infty(\Omega, C([0,T], V))$ and also,

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[0,T]} \|u(s)\|_V + E\left[\int_0^T \|u(s)\|^2_{D(A)} ds + \int_0^T \|Z(s)\|_V^2 ds\right] \\
&\le C\left\{\|g\|_{L^\infty(\Omega, L^1([0,T]))} + \|\xi\|^2_{L^\infty_{\mathscr{F}_T}(\Omega;V)}\right\},
\end{aligned}
\tag{5.8}
$$

where C is a constant depending on $\nu, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1$ and T.

Step 3. For a notational convenience, we now use the index N instead of N_k for all the relevant subsequences.

As $\cup_{N=1}^{\infty}\mathscr{L}_{\mathscr{F}}^2(0,T;P_NH)$ is dense in $\mathscr{L}_{\mathscr{F}}^2(0,T;H)$ and it can be checked that $\Psi=\Gamma$ on $\cup_{N=1}^{\infty}\mathscr{L}_{\mathscr{F}}^2(0,T;P_NH)$, by a density argument we have $\Psi=\Gamma$. Thus, to show that (u,Z) is a strong solution of the 2D BSNSE problem (2.4), we only need prove

$$\Psi(\cdot)=\Phi(\cdot,u,Z),\ a.s.. \tag{5.9}$$

For any $v\in L_{\mathscr{F}}^{\infty}(\Omega,C([0,T],V))$, define

$$r(t)=r(\omega,t):=\int_0^t (K+\frac{4}{\lambda}\|v(\omega,s)\|_V^2+K\rho^2(v(\omega,s)))\,ds,\quad (\omega,t)\in\Omega\times[0,T],$$

where the constant K comes from (3.14) in Lemma 3.2. Applying Itô formula to compute $e^{r(t)}\|u^N(t)\|^2$, we have

$$\begin{aligned}
&E\left[e^{r(t)}\|u^N(t)\|^2-e^{r(T)}\|u^N(T)\|^2\right]\\
&=E\bigg[\int_t^T e^{r(s)}\Big(\langle 2P_N\Phi(s,u^N(s),Z^N(s)),u^N(s)\rangle-\|Z^N(s)\|^2\\
&\quad-\big(K+\frac{4}{\lambda}\|v(s)\|_V^2+K\rho^2(v(s))\big)\|u^N(s)\|^2\Big)\,ds\bigg]\\
&=E\bigg[\int_t^T e^{r(s)}\bigg(2\langle\Phi(s,u^N(s),Z^N(s))-\Phi(s,v(s),Z(s)),u^N(s)-v(s)\rangle\\
&\quad-\|Z^N(s)-Z(s)\|^2-\bigg(K+\frac{4}{\lambda}\|v(s)\|_V^2+K\rho^2(v(s))\bigg)\|u^N(s)-v(s)\|^2\bigg)\,ds\bigg]\\
&\quad+E\bigg[\int_t^T e^{r(s)}\bigg(2\langle\Phi(s,u^N(s),Z^N(s))-\Phi(s,v(s),Z(s)),v(s)\rangle\\
&\quad+2\langle\Phi(s,v(s),Z(s)),u^N(s)\rangle-2\langle Z^N(s),Z(s)\rangle+\|Z(s)\|^2\\
&\quad-\bigg(K+\frac{4}{\lambda}\|v(s)\|_V^2+K\rho^2(v(s))\bigg)\big(2\langle u^N(s),v(s)\rangle-\|v(s)\|^2\big)\bigg)\,ds\bigg]\\
&\le E\bigg[\int_t^T e^{r(s)}\bigg(2\langle\Phi(s,u^N(s),Z^N(s))-\Phi(s,v(s),Z(s)),v(s)\rangle\\
&\quad+2\langle\Phi(s,v(s),Z(s)),u^N(s)\rangle-2\langle Z^N(s),Z(s)\rangle+\|Z(s)\|^2\\
&\quad-\bigg(K+\frac{4}{\lambda}\|v(s)\|_V^2+K\rho^2(v(s))\bigg)\big(2\langle u^N(s),v(s)\rangle-\|v(s)\|^2\big)\bigg)\,ds\bigg].
\end{aligned}$$

Letting $N\to\infty$, by Lemma 3.2 and the lower semicontinuity, we have for any

nonnegative $\varphi \in L^\infty(0,T)$,

$$
\begin{aligned}
& E\left[\int_0^T \varphi(t)\left(e^{r(t)}\|u(t)\|^2 - e^{r(T)}\|u(T)\|^2\right) dt\right] \\
\leq\ & \liminf_{N\to\infty} E\left[\int_0^T \varphi(t)\left(e^{r(t)}\|u^N(t)\|^2 - e^{r(T)}\|u^N(T)\|^2\right) dt\right] \\
\leq\ & E\bigg[\int_0^T \varphi(t)\bigg(\int_t^T e^{r(s)}\bigg(2\langle\Psi(s) - \Phi(s,v(s),Z(s)), v(s)\rangle \\
& \quad + 2\langle\Phi(s,v(s),Z(s)), u(s)\rangle - 2\langle Z(s), Z(s)\rangle + \|Z(s)\|^2 \\
& \quad - \left(K + \frac{4}{\lambda}\|v(s)\|_V^2 + K\rho^2(v(s))\right)\left(2\langle u(s), v(s)\rangle - \|v(s)\|^2\right)\bigg) ds\bigg) dt\bigg],
\end{aligned}
\tag{5.10}
$$

while Itô's formula yields

$$
\begin{aligned}
& E\left[e^{r(t)}\|u(t)\|^2 - e^{r(T)}\|u(T)\|^2\right] \\
=\ & E\bigg[\int_t^T e^{r(s)}\bigg(\langle 2\Psi(s), u(s)\rangle \\
& \quad - \|Z(s)\|^2 - \left(K + \frac{4}{\lambda}\|v(s)\|_V^2 + K\rho^2(v(s))\right)\|u(s)\|^2\bigg) ds\bigg].
\end{aligned}
\tag{5.11}
$$

By substituting (5.11) into (5.10), we get

$$
\begin{aligned}
E\bigg[\int_0^T \varphi(t)\bigg(\int_t^T e^{r(s)}\bigg(& 2\langle\Psi - \Phi(s,v(s),Z(s)), u(s) - v(s)\rangle \\
& - \left(K + \frac{4}{\lambda}\|v(s)\|_V^2 + K\rho^2(v(s))\right)\|u(s) - v(s)\|^2\bigg) ds\bigg) dt\bigg] \leq 0.
\end{aligned}
\tag{5.12}
$$

Take $v = u - \gamma\phi w$ for $\gamma > 0$, $w \in V$ and $\phi \in L^\infty(\Omega \times [0,T], \mathcal{P}, \mathbb{R})$. Then we divide by γ and let $\gamma \to 0$ to derive that

$$
E\bigg[\int_0^T \varphi(t)\bigg(\int_t^T e^{r(s)}\phi(s)\Big(2\langle\Psi - \Phi(s,u(s),Z(s)), w\rangle\Big) ds\bigg) dt\bigg] \leq 0. \tag{5.13}
$$

By the arbitrariness of φ,ϕ and w, we have

$$
\Gamma = \Psi = \Phi(\cdot, u, Z), \quad a.e. \text{ on } \Omega \times [0,T]
$$

In view of (5.7) and keeping in mind the fact $\bar{u} = u\ dt \times \mathbb{P}$-$a.e.$, we have

$$
u(t) = \xi + \int_t^T \Phi(s,u(s),Z(s))\, ds - \int_t^T Z(s)\, dW_s. \tag{5.14}
$$

Hence, by Remark 2 we conclude that (u, Z) is a strong solution to the 2D BSNSE problem (2.4).

Step 4. We shall prove the uniqueness. Suppose that there are two strong solutions $(u(\cdot), Z(\cdot))$ and $(v(\cdot), Y(\cdot))$ to the problem (2.4) corresponding to the same terminal data ξ. Then

$$\begin{aligned} u(t)-v(t) = &\int_t^T (-\nu Au(s)+\nu Av(s))\,ds + \int_t^T (B(u(s))-B(v(s)))\,ds \\ &+ \int_t^T (JZ(s)-JY(s))\,ds + \int_t^T (f(s,u(s),Z(s))-f(s,v(s),Y(s)))\,ds \\ &- \int_t^T (Z(s)-Y(s))\,dW_s, \quad t\in[0,T],\ a.s.. \end{aligned}$$

Define

$$R(t)=R(\omega,t)=\int_0^t (K+\frac{4}{\lambda}\|v(\omega,s)\|_V^2 + K\rho^2(v(\omega,s)))\,ds, \quad (\omega,t)\in\Omega\times[0,T].$$

Then in view of Lemma 3.2 and by Itô's formula (for instance, see [13, Theorem 4.2.5]), we have

$$\begin{aligned} &E_{\mathscr{F}_t}\left[e^{R(t)}\|u(t)-v(t)\|^2\right] \\ =&E_{\mathscr{F}_t}\Bigg[\int_t^T e^{R(s)}\Big(2\langle \Phi(s,u(s),Z(s))-\Phi(s,v(s),Y(s)),u(s)-v(s)\rangle \\ &-\|Z(s)-Y(s)\|^2-\left(\kappa+\frac{4}{\lambda}\|v(s)\|_V^2+K\rho^2(v(s))\right)\|u(s)-v(s)\|^2\Big)\,ds\Bigg] \\ \le&E_{\mathscr{F}_t}\Bigg[\int_t^T e^{R(s)}\Big(-\frac{\bar\lambda^2-1}{2\bar\lambda^2}\|Z(s)-Y(s)\|^2-\lambda\|u(s)-v(s)\|_V^2\Big)\,ds\Bigg]. \end{aligned} \tag{5.15}$$

Thus,

$$E_{\mathscr{F}_t}\Bigg[e^{R(t)}\|u(t)-v(t)\|^2+\int_t^T e^{R(s)}\Big(\frac{\bar\lambda^2-1}{2\bar\lambda^2}\|Z(s)-Y(s)\|^2+\lambda\|u(s)-v(s)\|_V^2\Big)ds\Bigg]\le 0,$$

which implies

$$\text{for any } t\in[0,T], u(t)-v(t)=0 \text{ in } H,\ a.s. \text{ and } E\left[\int_0^T \|Z(s)-Y(s)\|^2 ds\right]=0.$$

By the continuity of u and v, we have $\|(u-v, Z-Y)\|_{\mathscr{M}}=0$, from which we conclude that (u, Z) is only a modification of (v, Y) in $(L^2_{\mathscr{F}}(\Omega; C([0,T];V))\cap\mathscr{L}^2_{\mathscr{F}}(0,T;D(A)))\times\mathscr{L}^2_{\mathscr{F}}(0,T;V)$. We complete the proof. □

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